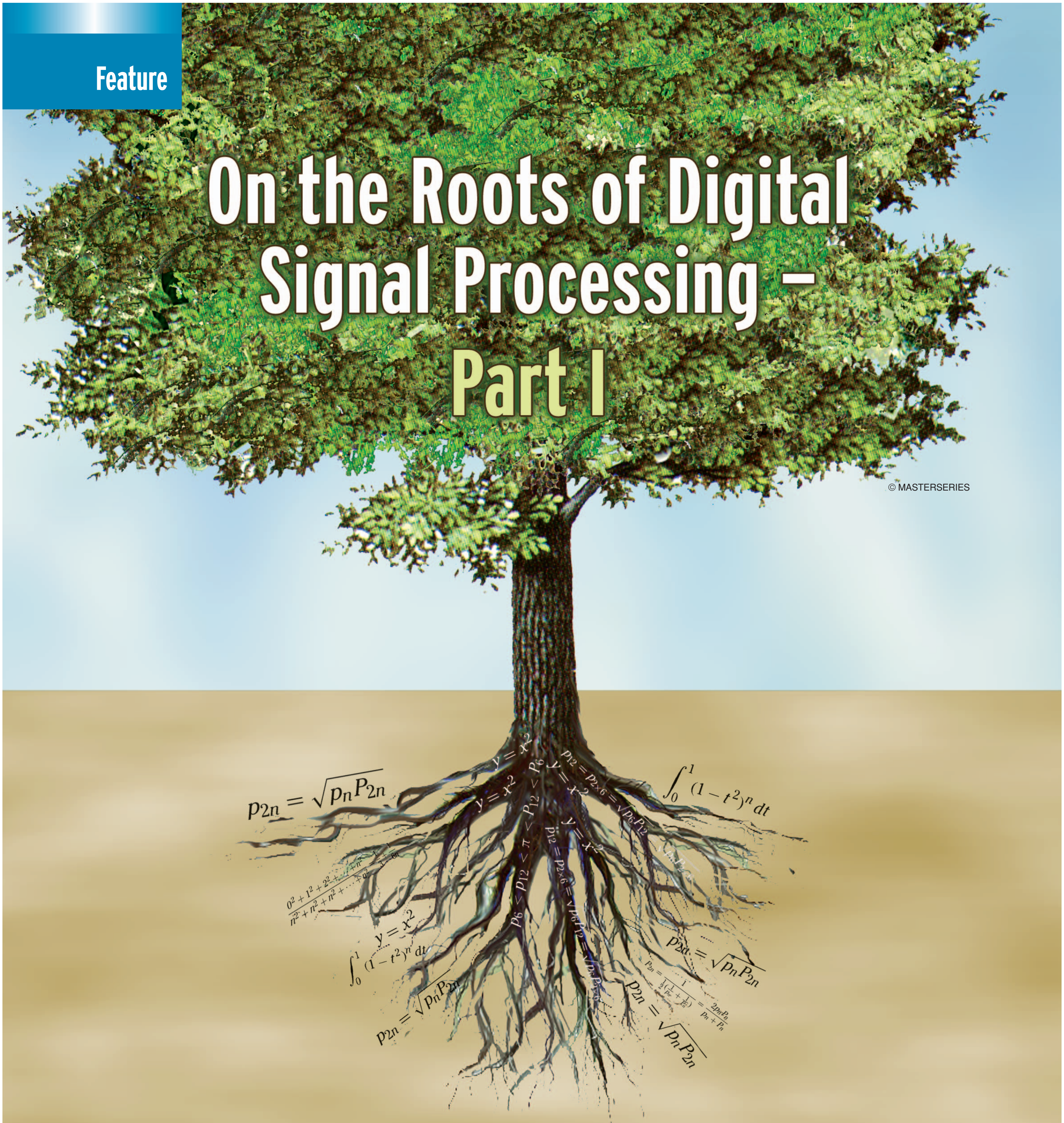
Feature

On the Roots of Digital Signal Processing – Part I



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Abstract

The roots of what we refer to today as digital signal processing are actually the roots of modern mathematics and to trace the evolution of DSP we need to go back to Newton, even to the great Archimedes of Syracuse. This two-part article will attempt in a not-so-rigorous exposition to outline the major historical developments that led to DSP.

I. Introduction

The evolution of the mathematics that pertains to DSP is dominated by six landmark events, namely, the development of the foundations of geometry during the Greek classical period from 700 to 100 B.C., the evolution of algebra by 800 A.D. in the Arab world, the emergence of calculus during the 1600s, the development of sophisticated numerical methods from the 17th to 19th century, the invention of the Fourier series during the early 1800s, the design and construction of computing machines, and the recent advancements in integrated-circuit technology.

This two-part article will highlight some of these landmark events and will attempt to show that the human need for discretized functions in the form of experimental data or numerical tables of one form or another led to a collection of mathematical principles that are very much a part of today's DSP. The need to mechanize the production of numerical tables through these mathematical principles led to the design of the difference engine by Babbage [1] and the construction of ENIAC, the first digital computer. In other words, contrary to popular belief, numerical methods that are akin to today's DSP and efforts to mechanize them gave birth to the modern digital computer and not the other way around.

II. What is DSP?

To be able to trace the roots of DSP, we need to trace the roots of the fundamental processes that make up DSP. Typically, we sample a continuous-time signal (or discretize some physical quantity), digitize it, process it, and then generate a processed version of the continuous signal (or physical quantity) through some interpolation scheme [2], as illustrated in Figure (1a) and (b). Thus, if we want to pinpoint the origins of DSP, we must find out when these processes began to emerge.

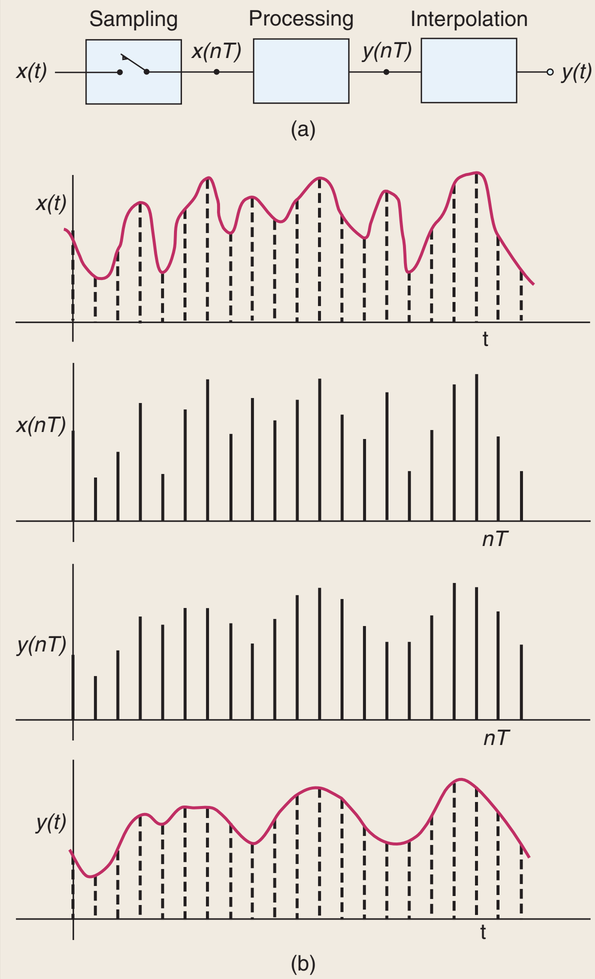

Figure 1. Sampling, processing, and interpolation of a continuous-time signal.

III. Archimedes

Everyone has heard of Archimedes of Syracuse (circa 287-212 B.C.) as the man who discovered the law pertaining to the weight of immersed bodies, the *Archimedes principle*. Apparently, he discovered this important physical law as he was contemplating on the best way to check the purity of a golden crown for king Hieron of Syracuse while he was taking a bath. The king had reason to believe that the goldsmith who made the crown kept some of the king's gold for himself and made up the difference by adding an equal weight of silver. When the solution of the puzzle dawned on Archimedes, he ran out into the street naked shouting *eurika*, *eurika*, a word that is understood by Greeks and non-Greeks alike to mean *I discovered*.[1] However, he did much more than that. He contributed quite substantially to the theory of mechanics and mathematics, and wrote books on these subjects (see [3]–[5] for more details about Archimedes and all the other great mathematicians mentioned in this article). The one of his great achievements that bears an ancestral relation to DSP is his method of calculating the perimeter ($\pi\varepsilon\rho\iota\mu\varepsilon\tau\rho o\varsigma$ in Greek) of a circle with a diameter of one unit, which happens to be equal to π. The connection between the perimeter of a circle to the underlying principles of DSP

[1] The answer to the riddle turned out to be amazingly simple: weigh the crown against an equal weight of pure gold with the pans of the scales immersed in water and watch the beam of the scales!

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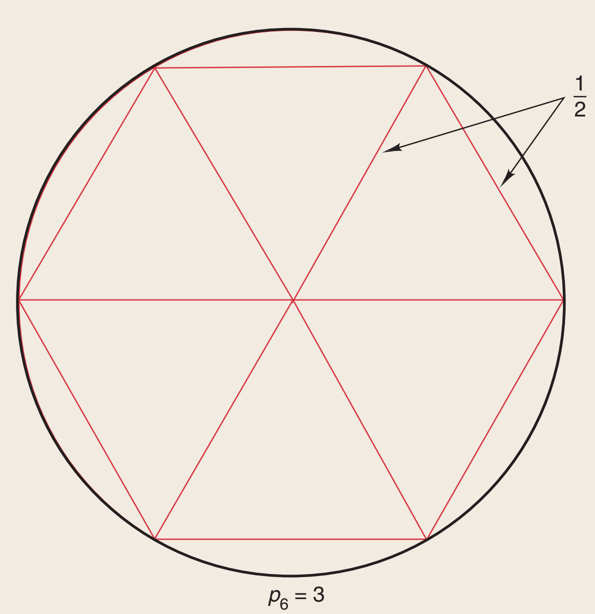

Figure 2. Hexagon inscribed in a circle of diameter of one unit.

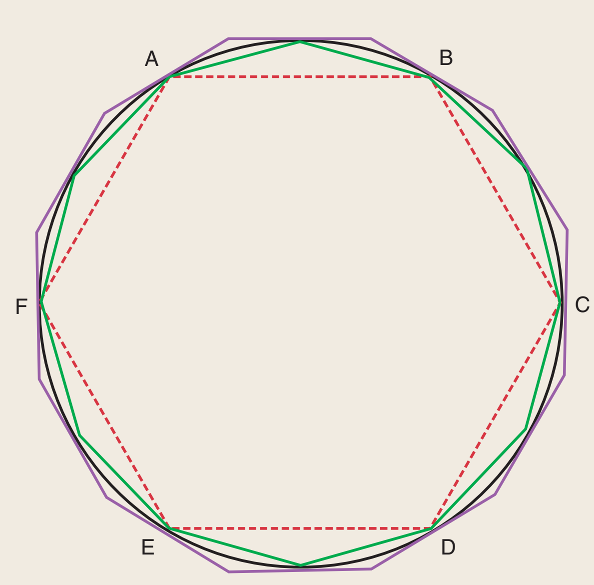

Figure 4. Circle of diameter of one unit sandwiched between 12-sided polygons.

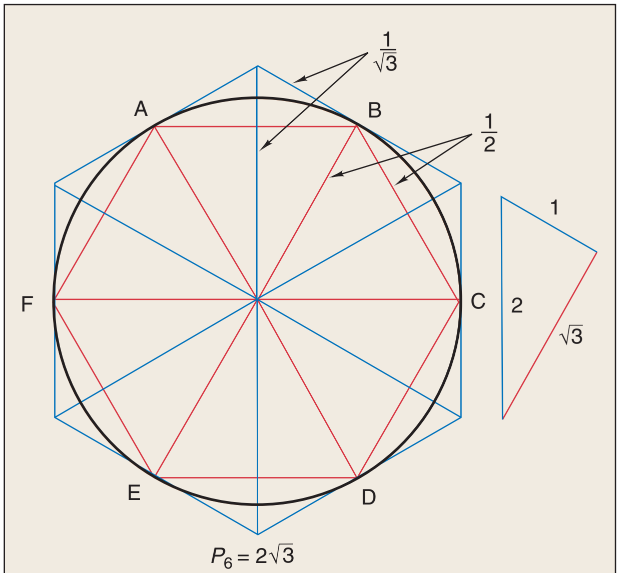

Figure 3. Hexagon circumscribing a circle of diameter of one unit.

can be revealed by looking at Archimedes' method from the perspective of a DSP practitioner.

He inscribed a hexagon in a circle of diameter of one unit and divided the hexagon into six equilateral triangles as illustrated in Figure 2. Since the diameter is one unit long, the sides of each triangle are half a unit long. Noting that the shortest trajectory between two points is a straight line, Archimedes concluded that the perimeter of the circle, π, is larger than 3, the perimeter of the hexagon, since each side of the hexagon is half a unit long. If we denote the perimeter of the hexagon as p_6, then $p_6 = 3 < \pi$.

He then circumscribed the circle by another hexagon as shown in Figure 3. This he did by drawing tangents to the circle at vertices A, B, C, ... of the inside hexagon by using just a straight edge.[2] Simple geometry, which was the forte of the Greeks, gives the perimeter of the larger hexagon as $P_6 = 6 \times 1/\sqrt{3} = 2\sqrt{3} = 3.4641$ and since the outside hexagon is obviously larger than the circle, we have $\pi < P_6 = 3.4641$. What Archimedes did in terms of today's terminology is to find lower and upper bounds for the perimeter of a circle, i.e.,

$$p_6 = 3 < \pi < 3.4641 = P_6 \tag{1}$$

Most of us would have stopped at this point. However, Archimedes' mathematical genius pushed him on to the next step. He doubled the sides of the inside hexagon by bisecting each of its sides (using just a compass) to obtain a regular 12-sided polygon (dodecagon in Greek) shown in green in Figure 4. After that he generated a regular 12-sided polygon that circumscribed the circle shown in purple in Figure 4 by simply drawing tangents at the vertices of the inside 12-sided polygon. Through geometry, he found out that the perimeter of the outside polygon, P_{12}, is given by the *harmonic mean* of p_6 and P_6, which is defined as the reciprocal of the arithmetic mean of the reciprocals of the two numbers, i.e.,

[2]The ancients considered it a virtue to use the minimum number of tools for geometrical constructions!

$$P_{12} = P_{2\times 6} = \frac{1}{\frac{1}{2}(\frac{1}{p_6} + \frac{1}{P_6})} = \frac{2p_6P_6}{p_6 + P_6} \quad (2a)$$

and p_{12} is given by the *geometric mean* of p_6 and P_{12}, i.e.,

$$p_{12} = p_{2\times 6} = \sqrt{p_6P_{12}} = \sqrt{p_6P_{2\times 6}} \quad (2b)$$

It should be clarified here that Archimedes' exposition of these principles was in terms of geometry. Algebra did not emerge as a subject of study until much later. Although it is not known who invented it or how it emerged, an Arab mathematician by the name of al-Khwarizmi (c. 780–850 A.D.) wrote a treatise on the subject, which greatly influenced European mathematics [4], [5]. The word "algebra" originates from "al-jabr" the first word in the title of al-Kwarizmi's treatise and "algorithm" is actually a Latin corruption of his name.[3]

Using Eqs. (2a) and (2b), we get $P_{12} = 3.2154$ and $p_{12} = 3.1058$. Therefore, from Eq. (1) we have

$$3 < 3.1058 < \pi < 3.2154 < 3.4641$$

or

$$p_6 < p_{12} < \pi < P_{12} < P_6$$

Obviously, the inside and outside 12-sided polygons provide tighter bounds on the perimeter of the circle.

A man of genius as he was, Archimedes noted that the relations in Eqs. (2a) and (2b) readily extend to 24-, 48-, and 96-sided polygons. In fact, they hold true in general for a $2n$-sided polygon, i.e.,

$$P_{2n} = \frac{1}{\frac{1}{2}(\frac{1}{p_n} + \frac{1}{P_n})} = \frac{2p_nP_n}{p_n + P_n} \quad (3a)$$

and

$$p_{2n} = \sqrt{p_nP_{2n}} \quad (3b)$$

(see [5] for details). Using these relations, the numerical values of the lower and upper bounds of π, given in Table I can be obtained. Evidently, these bounds are, in effect, discrete functions of n, as depicted in Figure 5, and their averages are successive approximations for the perimeter of the circle. In a way, Archimedes subjected the perimeter of the circle to a discretization process and interpolated the results to estimate the perimeter of the circle.

The average value of π on the 5th iteration of what has been referred to as the *Archimedes algorithm* works out to be 3.1419 which entails an error of about 0.008%.

Table 1.
Lower and Upper Bounds on π.

Iter.	No. of sides	Lower bound	Upper bound
1	6	3.0000	3.4641
2	12	3.1058	3.2154
3	24	3.1326	3.1597
4	48	3.1394	3.1461
5	96	3.1410	3.1427

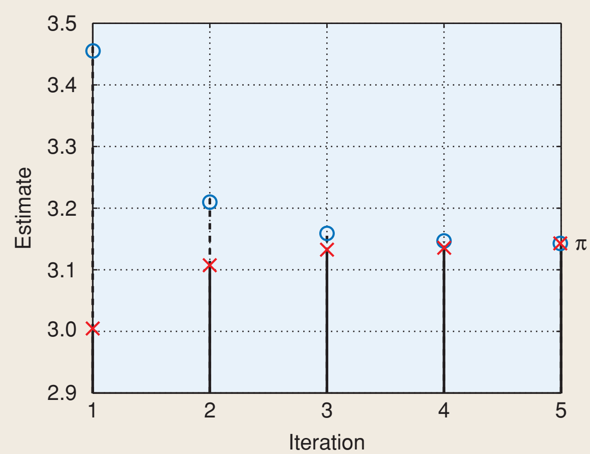

Figure 5. Perimeters of the inside and outside polygons.

The usual rational approximation for π, namely, $22/7 = 3.1429$, is sometimes referred to as the Archimedean π but it is not known whether he had anything to do with it. However, he had a great deal to do with the formula that gives the area of a circle, i.e., πr^2. Once he subdivided the inside and outside polygons into isosceles triangles, it was an easy matter to obtain an approximation for the area of the circle since the ancients knew how to handle triangles.

Whether he was calculating the perimeter or the area of the circle, Archimedes stopped with 96-sided polygons, i.e., on the 5th iteration of the algorithm. Continuing the algorithm for 16 iterations (393,216-sided polygons) yields the value of π to a precision better than 1 part in 10^{10} according to MATLAB®. Had he mentioned even briefly that the perimeters of the inside and outside polygons, or their average, would coincide with the perimeter of the circle if n were increased to infinity, he would have introduced the concept of infinity and that of the limit but these principles had to wait for some 2000 years before they could become part of modern European mathematics during the 1600s. This trail of events will be picked up in the next section.

[3] Al-Kwarizmi also wrote a short treatise on arithmetics in which he used Hindu numerals, now commonly referred to as Arabic numerals, but that's another story [4], [5].

It should be mentioned that interest in approximating the circle in terms of polygons resurfaced quite independently in ancient China where a certain Liu Hui (c. 220–280 A.D.) calculated the bounds

$$3.141024 < \pi < 3.142904$$

using 384-sided polygons and deduced the approximation 3.14159 for π using 3072-sided polygons. Later on, Tsu Chung-Chi (430–501) deduced the more precise bounds

$$3.1415926 < \pi < 3.1415927$$

and referred to 22/7 as an *inaccurate* value and 355/113 as an *accurate* value of π [4], [5]. Amazingly, the latter rational approximation entails an error of about 8×10^{-6}%!

IV. The Renaissance of Mathematics

Renewed interest in mathematics, in the sciences in general, and in the work of Archimedes in particular emerged in Europe during the early 1600s. John Wallis (1616–1703) and James Gregory (1638–1675), the first an Englishman and the second a Scot mathematician, developed techniques for finding the areas of various geometric figures by extending the approach of Archimedes.

Wallis considered the area under the parabola

$$y = x^2$$

to be made up of a series of rectangles [5] as depicted in Figure 6(a), each of base ε. He noted that

$$\text{Area abcfa} \approx (k\varepsilon)^2 \cdot \varepsilon = k^2\varepsilon^3$$

and that

$$\text{Area abdea} = (n\varepsilon)^2 \cdot \varepsilon = n^2\varepsilon^3$$

Therefore, the area under the parabola, A_P, is related to the area of the rectangle ABCDA, A_R, by the approximation

$$A_P \approx \frac{(0^2 + 1^2 + 2^2 + \cdots + n^2)\varepsilon^3}{(n^2 + n^2 + n^2 + \cdots + n^2)\varepsilon^3} \cdot A_R \tag{4}$$

He then observed that

$$\begin{aligned} \frac{0^2 + 1^2}{1^2 + 1^2} &= \frac{1}{2} = \frac{1}{3} + \frac{1}{6} \\ \frac{0^2 + 1^2 + 2^2}{2^2 + 2^2 + 2^2} &= \frac{5}{12} = \frac{1}{3} + \frac{1}{12} \\ \frac{0^2 + 1^2 + 2^2 + 3^2}{3^2 + 3^2 + 3^2 + 3^2} &= \frac{7}{18} = \frac{1}{3} + \frac{1}{18} \\ &\vdots \end{aligned}$$

and by applying the principle of induction, he concluded that

$$\frac{0^2 + 1^2 + 2^2 + \cdots + n^2}{n^2 + n^2 + n^2 + \cdots + n^2} = \frac{1}{3} + \frac{1}{6n} \tag{5}$$

He no doubt noticed that as the number of rectangles increased and the base of each rectangle reduced, the sum of the areas of the rectangles tended to get closer and closer to the area under of the parabola. He then took a giant step forward by making the base of each rectangle infinitesimally small and to compensate for that he made the number of rectangles infinitely large, and by using Eqs. (4) and (5), he concluded that

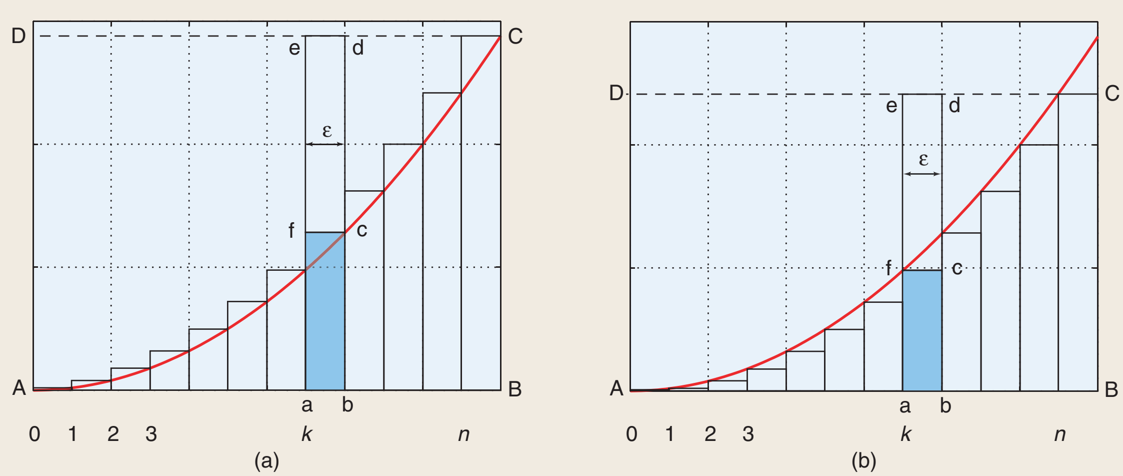

Figure 6. Wallis' geometrical construction.

$$\begin{aligned} A_P &= \lim_{n\to\infty} \frac{(0^2+1^2+2^2+\cdots+n^2)\varepsilon^3}{(n^2+n^2+n^2+\cdots+n^2)\varepsilon^3}\cdot A_R \\ &= \lim_{n\to\infty}\left(\frac{1}{3}+\frac{1}{6n}\right)A_R \\ &= \frac{1}{3}A_R \end{aligned}$$

What Wallis did, in effect, was to discretize the parabola by circumscribing it by a piecewise-constant discrete function in the same way as Archimedes had discretized the circle by circumscribing it by an n-sided polygon. Wallis could also have achieved the same result by inscribing a piecewise-constant discrete function to the parabola as depicted in Figure 6(b) to complete the analogy with Archimedes' approach.

In one sweeping scoop, Wallis introduce the concept of infinity, he discovered the principle of the limit which so eluded the ancients,[4] and laid down the basics of integration as we know it today. He also proposed the symbol for infinity (∞) and coined the word *interpolation* according to the historians.

From the perspective of the DSP practitioner, by discretizing the parabola Wallis introduced the principle of sampling and, in fact, the representation in Figure 6(b) is what we commonly refer to in the DSP literature as a 'sample-and-hold' operation. If the function in Figure 6(b) were a signal, then the rectangles in Figure 6(a) or (b) would be pulses which would become impulses as $\varepsilon \to 0$. In other words, Wallis' representation is analogous to sampling a signal by means of impulse modulation, as illustrated in Figure 7 (see Chap. 6 in [2]).

Expert in finding areas as he was, Wallis tabulate the values of the areas

$$\int_0^1 (1-t^2)^n\, dt \qquad (6)$$

(in today's notation) for certain integer values of n and showed that [4], [5]

$$\begin{aligned} \int_0^1 (1-t^2)^{\frac{1}{2}}\, dt &= \frac{\pi}{2} \\ &= \lim_{N\to\infty} \frac{2^2\cdot 4^2\cdot 6^2\cdots(N-1)^2}{1^2\cdot 3^2\cdot 5^2\cdot 7^2\cdots N} \\ &= \lim_{N\to\infty} \frac{2^2\cdot 4^2\cdot 6^2\cdot 8^2\cdots N}{1^2\cdot 3^2\cdot 5^2\cdot 7^2\cdots(N-1)^2} \end{aligned}$$

which is known as *Wallis' formula* for π. The formula is largely of historical interest since its convergence is rather slow, as can be easily verified by using MATLAB®.[5]

Gregory extended Archimedes' algorithm for the evaluation of the area of a circle to the evaluation of the area of an ellipse. He inscribed a triangle of area a_0 in the ellipse and circumscribed the ellipse by a quadrilateral (4-sided polygon) of area A_0, as illustrated in Figure 8. By successively doubling the numbers of sides in the inscribed and circumscribing polygons, he generated the sequence $a_0,\ A_0,\ a_1,\ A_1 \ldots a_n,\ A_n,\ \ldots$ and showed that a_n is the geometric mean of a_{n-1} and A_{n-1} whereas A_n is the harmonic mean of A_{n-1} and a_n , i.e.,

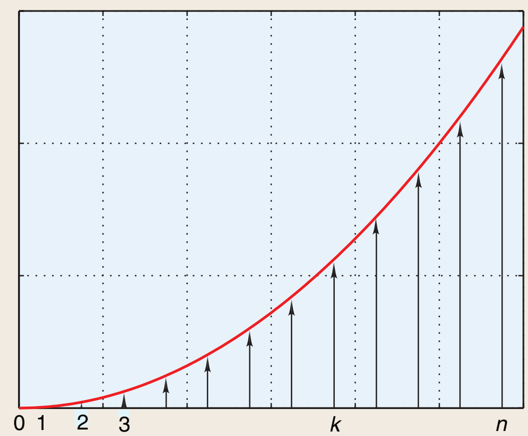

Figure 7. Decomposition of $y = x^2$ into an infinite series of impulse functions.

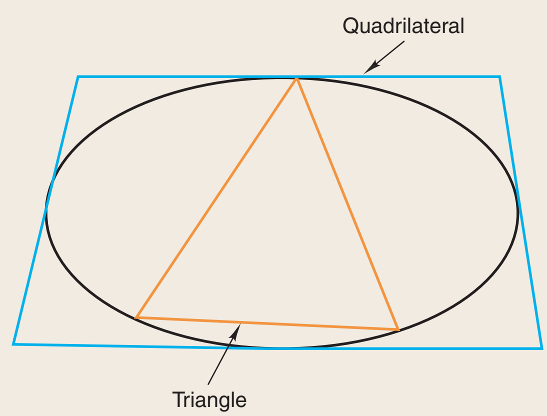

Figure 8. Gregory's approach to the area of an ellipse.

[4]Not knowing the principle of the limit, Zeno of Elea (c. 490–425 B.C.) described a number of paradoxes that contradicted observation [4], [5]; for example, that an arrow would never reach its target because it would have to cross the midpoint of its flight path, and having done so, it would have to cross the midpoint of the remaining distance, and so on; since a small distance would always remain, the arrow would never reach its target!

[5]Wallis had many talents in addition to mathematics. During the Civil War in England between the Royalists and the Parliamentarians, he used his skills in cryptography to decode Royalist messages for the Parliamentarians and, surprisingly, when the monarchy was restored some years later on, he was appointed a royal chaplain for Charles II [3].

$$a_n = \sqrt{a_{n-1}A_{n-1}} \quad \text{and} \quad A_n = \frac{2A_{n-1}a_n}{A_{n-1}+a_n}$$

as in Eqs. (3a) and (3b).[6] He then constructed two sequences, namely,

$$a_0, a_1, \ldots a_n, \ldots \quad \text{and} \quad A_0, A_1 \ldots A_n, \ldots$$

which, in his terminology, would *converge* to the area of the ellipse if n were made infinitely large. According to historians, Gregory is the first man to have used the word "converge" in a mathematical sense.

Gregory is also known for his work on infinite series and, in fact, the series

$$\int_0^x \frac{1}{1+x^2}\,dx = \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

is known as Gregory's series. He also discovered the Taylor series some 44 years before it was first published by Brook Taylor (1685–1731) [3], [4]. Apparently, according to some fairly recent findings, Gregory wrote the series on the back side of a letter he received but for some reason he never published that great formula.[7]

The work of Wallis was continued by Newton (1642–1727) who replaced the fixed upper limit of unity in Eq. (6) by x and following Wallis' methodology, he was able to obtained the results

$$\int_0^x (1-t^2)\,dt = x - \frac{1}{3}x^3$$
$$\int_0^x (1-t^2)^2\,dt = x - \frac{2}{3}x^3 + \frac{1}{5}x^5$$
$$\int_0^x (1-t^2)^3\,dt = x - \frac{3}{3}x^3 + \frac{3}{5}x^5 - \frac{1}{7}x^7$$

etc. Then, through laborious interpolation, he also found out that

Table 2.
Pascal Triangle.

					1					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1
⋮					⋮					⋮

$$\int_0^x (1-t^2)^{\frac{1}{2}}\,dt = x - \frac{\frac{1}{2}}{3}x^3 - \frac{\frac{1}{8}}{5}x^5 - \cdots$$

The amazing regularity of his solutions led him to conclude that

$$\int_0^x (1-t^2)^k\,dt = x - \frac{1}{3}\binom{k}{1}x^3 + \frac{1}{5}\binom{k}{2}x^5 - \cdots + \frac{1}{2n+1}\binom{k}{n}x^{2n+1} - \cdots$$

where

$$\binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!}$$

and by applying the method of tangents to both sides of the equation, i.e., by differentiating the two sides, he discovered the series

$$(1-x^2)^k = 1 - \binom{k}{1}x^2 + \binom{k}{2}x^4 - \cdots + \binom{k}{n}x^{2n} - \cdots$$

If we let, $-x^2 \to x$, the binomial theorem in its standard form is revealed, i.e.,

$$(1+x)^k = 1 + \binom{k}{1}x + \binom{k}{2}x^2 + \cdots + \binom{k}{n}x^n + \cdots \tag{7}$$

It should be mentioned that the expansion in Eq. (7) for positive integers was known long before the times of Newton in terms of the so-called *Pascal triangle*, shown in Table II, which, in turn, was known long before the times of Pascal according to the historians. Apparently, it first appeared in a treatise written by a Chinese mathematician by the name of Chu Shih-chieh (c. 1260–1320) and first showed up in print in Europe on the title page of a book on arithmetic written by Peter Apian (1495–1552). The triangle is named after Pascal (1623–1662) largely on the basis of his systematic investigation of the triangle's inherent relations, not because he discovered it [5].

The binomial theorem was investigated by many others after Newton including the great Gauss (1777–1855) who generalized its application to arbitrary rational

[6]The similarity of these formulas to those in Eqs. (3a) and (3b) should not be surprising! After all, a circle is a particular ellipse.
[7]Unfortunately for mathematics, Gregory had a stroke at the age of 36 and died a few days later [3] leaving much of his work unpublished.

Renewed interest in mathematics, in the sciences in general, and in the work of Archimedes in particular emerged in Europe during the early 1600s.

values of n.[8] On the other hand, work by Euler (1707–1783), Gauss, Cauchy (1789–1857), and Laurent (1813–1854) on complex numbers, complex variables, and functions of a complex variable has shown that the binomial theorem is also applicable to the case where x is a complex variable.

If we let $x = z^{-1}$ in the binomial series of Eq. (7), where z is a complex variable, we get

$$(1+z^{-1})^k = 1 + \binom{k}{1}z^{-1} + \binom{k}{2}z^{-2} + \cdots + \binom{k}{n}z^{-n} + \cdots$$

which is referred to in DSP literature as the z transform of right-sided signal

$$x(nT) = u(nT)\binom{k}{n}$$

where $u(nT)$ is the discrete-time unit-step function.

The inverse z transform can often be deduced by simply finding the coefficient of z^{-n} in a binomial series expansion of the function, as will now be demonstrated. Consider the z transform

$$X(z) = \frac{Kz^m}{(z-w)^k}$$

where m and k are integers, and K and w are real or complex constants (see Example 3.4 in [2]). According to Eq. (7), we can write

$$\begin{aligned} X(z) &= Kz^{m-k}[1+(-wz^{-1})]^{-k} \\ &= Kz^{m-k}\left[1+\binom{-k}{1}(-wz^{-1}) + \binom{-k}{2}(-wz^{-1})^2 + \cdots + \binom{-k}{n}(-wz^{-1})^n + \cdots\right] \end{aligned}$$

If we let $n = n' + m - k$ and then replace n' by n, we have

Table 3.
Standard z Transforms.

$x(nT)$	$X(z)$
$u(nT)$	$\frac{z}{z-1}$
$u(nT-kT)K$	$\frac{Kz^{-(k-1)}}{z-1}$
$u(nT)Kw^n$	$\frac{Kz}{z-w}$
$u(nT-kT)Kw^{n-1}$	$\frac{K(z/w)^{-(k-1)}}{z-w}$
$u(nT)e^{-\alpha nT}$	$\frac{z}{z-e^{-\alpha T}}$
$u(nT)nT$	$\frac{Tz}{(z-1)^2}$
$u(nT)nTe^{-\alpha nT}$	$\frac{Te^{-\alpha T}z}{(z-e^{-\alpha T})^2}$

$$X(z) = \sum_{n=-\infty}^{\infty}\left\{Ku[(n+m-k)T] \cdot \frac{(-k)(-k-1)\cdots(-n-m+1)(-w)^{n+m-k}}{(n+m-k)!}\right\}\cdot z^{-n}$$

Now noting that

$$(-k)(-k-1)\cdots(-n-m+1) = \frac{(-n-m+1)!}{(k-1)!} \times (-1)^{n+m-k}$$

the inverse z transform, which is the coefficient of z^{-n}, can be obtained as

$$\begin{aligned} x(nT) &= \mathcal{Z}^{-1}\left[\frac{Kz^m}{(z-w)^k}\right] \\ &= Ku[(n+m-k)T]\frac{(n+m-1)!w^{n+m-k}}{(k-1)!(n+m-k)!} \end{aligned}$$

By assigning specific values to constants k, K, and m, an entire table of z transforms can be constructed, as shown in Table III, by using nothing more than Newton's binomial theorem.

While Wallis and Gregory were developing methods for finding the areas of geometrical figures, others were investigating methods for finding the slopes of curves. These were known as the methods quadrature and the methods of tangents, respectively, in those days, i.e.,

[8]Some say that Gauss 'discovered' the binomial theorem without knowledge of Newton's work at the age of 15.

Interest in the movement of the planets and other celestial bodies grew very strong in those days after the discoveries of Galileo during the early 1600s, and the astronomers of the time needed to fit curves to their measured data.

integration and differentiation, in modern language. It was also known that there was a relation between the two types of methods and, in fact, Gregory actually proved that there was an inverse relation between them. However, it took people like Newton and Leibniz during the late 1600s to rationalize all these principles into the unified field of study we know today as calculus.

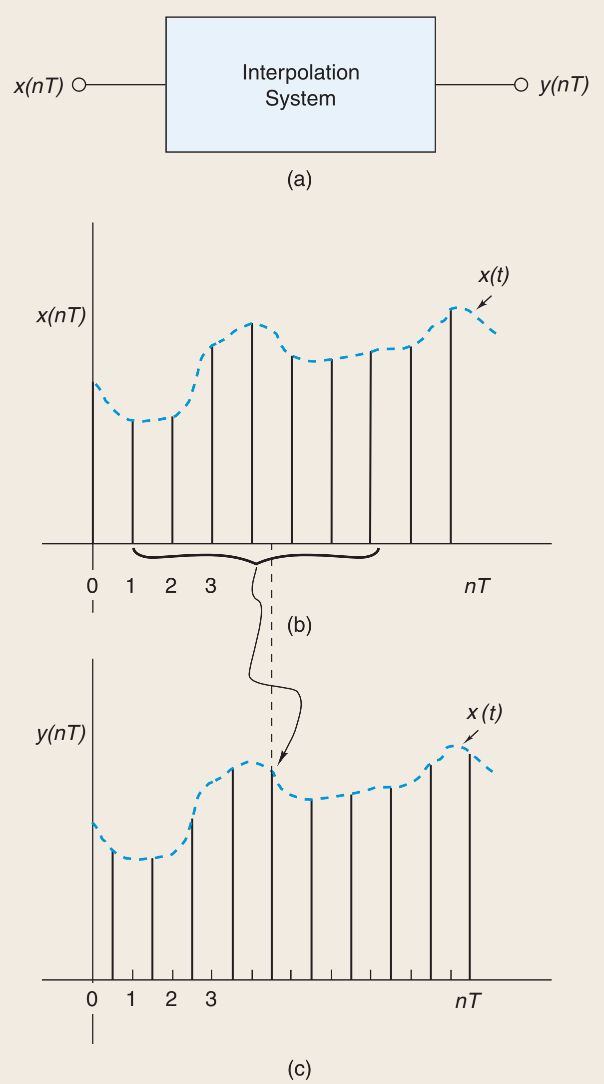

Figure 9. Interpolation process: (a) Interpolating system, (b) excitation, (c) response.

Newton, as is very well known, made numerous other contributions to science, e.g., he formulated his laws of motion, proposed a theory of gravitation that explained the dynamical interactions among heavenly bodies, hypothesized for the first time that white light is a mixture of many different types of rays, constructed a reflecting telescope, etc. What is less well known is that he also served as the Master of the British Mint by royal appointment during his later years. In this capacity, he supervised the production of new coins which were much more difficult to counterfeit, and was also in charge of prosecuting counterfeiters!

Interest in the movement of the planets and other celestial bodies grew very strong in those days after the discoveries of Galileo during the early 1600s, and the astronomers of the time needed to fit curves to their measured data. If Archimedes and Wallis had nothing to do with sampling, then the astronomers of the middle ages had a great deal to do with it because their measurements were discrete-time functions in today's DSP terminology. And to convert their measured data into formulas that described the continuous trajectories of the celestial bodies, powerful interpolation methods were required. This problem was explored by James Stirling (1692–1770), Joseph-Louis Lagrange (1736–1813), and Wilhelm Bessel (1784–1846), to name just three, who proposed numerical interpolation formulas that could be applied to functions in tabular form.

V. Stirling's Interpolation Formula

Among the many interpolation formulas, that of Stirling is of particular interest because, as will be demonstrated below, it can be used to design digital filters that can perform interpolation, differentiation, and integration.

If the values of $x(nT)$ are known for $n = 0, 1, 2, \ldots,$ then the value of $x(nT + pT)$ for some fraction p in the range $0 < p < 1$ can be determined by using Stirling's interpolation formula

$$\begin{aligned}
x(nT + pT) &\\
= \left[1 + \frac{p^2}{2!}\delta^2 + \frac{p^2(p^2-1)}{4!}\delta^4 + \cdots\right] & x(nT)\\
+ \frac{p}{2}\left[\delta x\left(nT - \frac{1}{2}T\right) + \delta x\left(nT + \frac{1}{2}T\right)\right] &\\
+ \frac{p(p^2-1)}{2(3!)}\left[\delta^3 x\left(nT - \frac{1}{2}T\right) + \delta^3 x\left(nT + \frac{1}{2}T\right)\right] &\\
+ \frac{p(p^2-1)(p^2-2^2)}{2(5!)}\left[\delta^5 x\left(nT - \frac{1}{2}T\right)\right. &\\
\left. + \delta^5 x\left(nT + \frac{1}{2}T\right)\right] + \cdots & \qquad (8)
\end{aligned}$$

where

$$\delta x\left(nT+\frac{1}{2}T\right)=x(nT+T)-x(nT) \tag{9}$$

denotes the central difference of $x(nT+\frac{1}{2}T)$.

By differentiating or integrating Eq. (8), formulas for numerical differentiation or integration can be deduced (see [2] for further details).

Neglecting differences of order 6 or higher and letting $p=1/2$ in Eq. (8), and then eliminating the central differences using Eq. (9), we get

$$y(nT)=x\left(nT+\frac{1}{2}T\right)=\sum_{i=-3}^{3}h(iT)x(nT-iT)$$

where coefficients $h(iT)$ are given in Table IV. This is recognized as the difference equation of a nonrecursive (also known as an FIR) discrete-time system as illustrated in Figure 9.

Interpolation is a process that would fit a *smooth* curve through a number of sample points as illustrated in Figure 9(b) and, consequently, one would expect interpolation to be akin to lowpass filtering. To check this out, we can represent the interpolation system of Figure 9(a) by the transfer function

$$H(z)=\frac{Y(z)}{X(z)}=\sum_{k=-3}^{3}h(iT)z^{-k}$$

Hence its frequency response, amplitude response, and phase response can be deduced as

$$H(e^{j\omega T})=\sum_{i=-3}^{3}h(iT)e^{-jk\omega T}$$

$$M(\omega)=\left|\sum_{i=-3}^{3}h(iT)e^{-jk\omega T}\right|$$

and

Table 4.
Coefficients $h(iT)$.

i	$h(iT)$
−3	−5.859375E−3
−2	4.687500E−2
−1	−1.855469E−1
0	7.031250E−1
1	4.980469E−1
2	−6.250000E−2
3	5.859375E−3

With the invention of the printing press by Johann Gutenberg in 1450, numerical tables of all forms began to be published and interest in generating numerical tables, i.e., discretized functions, became very important for business, science, navigation, and so on.

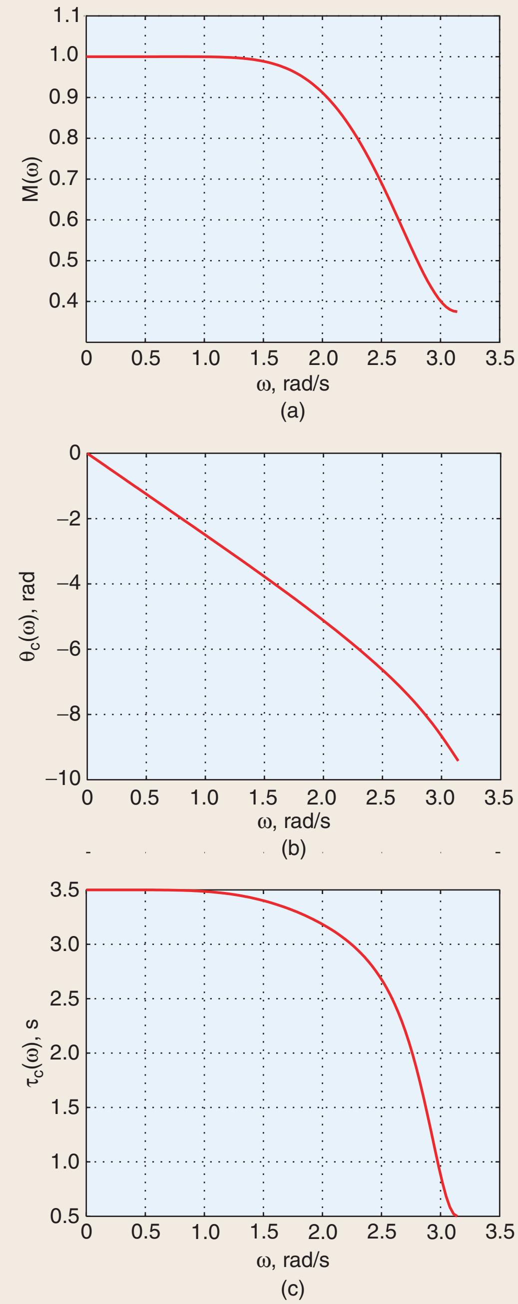

Figure 10. Frequency response of interpolating system: (a) Amplitude response, (b) phase response, (c) delay characteristic.

$$\theta(\omega) = \arg \sum_{i=-3}^{3} h(iT) e^{-jk\omega T}$$

respectively. Using the numerical values of $h(iT)$ given in Table IV, the amplitude response of the interpolation system shown in Figure 10(a) can be obtained which is hardly different from the response of a lowpass filter.

The use of Stirling's formula for the design of nonrecursive filters, just like the Fourier series method, yields noncausal filters (see Chap. 9 of [2]) but by delaying the impulse response by a period $(N-1)T/2$, where N is the filter length, a causal filter can be obtained. Applying this correction to the interpolator design yields a modified phase response $\theta_c(\omega) = -3\omega + \theta(\omega)$. The phase response of the causal interpolator and the corresponding delay characteristic, $\tau_c(\omega) = -d\theta_c(\omega)/d\omega$, are depicted in Figure 10(b) and (c), respectively. As can be seen the phase response is linear and the delay characteristic is approximately flat with respect to the passband of the interpolator. And most importantly, the interpolator was designed using concepts proposed some 250 years ago.

VI. Discretization of Functions

With the invention of the printing press by Johann Gutenberg in 1450, numerical tables of all forms began to be published and interest in generating numerical tables, i.e., discretized functions, became very important for business, science, navigation, and so on. Just like today, the necessary computations were carried out by computers except that in those days the computers were all human! Apparently, the word computer did not acquired its modern inanimate meaning until fairly recently, after the invention of the modern digital computer, according to Swade [1]. Consequently, published tables were full of erroneous entries. To circumvent this problem, inventors of all types from Pascal to Babbage were trying to built calculating machines. Part II of this article, to be published in another issue of the CAS magazine, will deal with some of the efforts to mechanize computing and the relation of these efforts to DSP. The second part will also examine the contributions of a group of French mathematicians, namely, Laplace, Fourier, Poisson, and Laurent, whose breakthroughs during the late 1700s and early 1800s became the backbone of spectral analysis. Part II will also deal with the evolution of DSP during the first half of the twentieth century and conclude with the beginnings of modern DSP during the late sixties and early seventies.

VII. Conclusions

It has been demonstrated that the basic processes of DSP, namely, discretization (or sampling) and interpolation have been a part of mathematics in one form or another since classical times, and mathematical discoveries made since the early 1600s are very much a part of the toolbox of a modern DSP practitioner. Part II of this article will demonstrate that the need to construct accurate discretized functions in the form of numerical tables efficiently led to the difference engine of Babbage during the 1800s and, in due course, to the modern digital computer during the late 1940s. The application of the early digital computers for the analysis of signals and systems marked the beginnings of the modern era of DSP.

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